Lesson 2.5

Determinants and Multiplicative Inverses of Matrices.

Key Concept — **Second-Order Determinant**

For Your FOLDABLE

Words The value of a second-order determinant is the difference of the products of the two diagonals.

Symbols $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 4 & 5 \\ -3 & 6 \end{vmatrix} = 4(6) - (-3)(5) = 39$

EXAMPLE 1 Second-Order Determinant

Evaluate the determinant $\begin{vmatrix} 6 & 4 \\ -1 & 0 \end{vmatrix}$.

$$\begin{vmatrix} 6 & 4 \\ -1 & 0 \end{vmatrix} = 6(0) - 4(-1)$$ Definition of determinant

$$= 0 + 4$$ Multiply.

$$= 4$$ Simplify.

Answer: 4

Third – Order Determinant

Key Concept — **Diagonal Rule**

For Your FOLDABLE

Step 1 Rewrite the first two columns to the right of the determinant.

Step 2 Draw diagonals, beginning with the upper left-hand element. Multiply the elements in each diagonal. Repeat the process, beginning with the upper right-hand element.

Step 3 Find the sum of the products of the elements in each set of diagonals.

Step 4 Subtract the second sum from the first sum.

EXAMPLE 2 Use Diagonals

Evaluate $\begin{vmatrix} 3 & -2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{vmatrix}$ **using diagonals.**

Step 1 Rewrite the first 2 columns to the right of the determinant.

$$\begin{vmatrix} 3 & -2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{vmatrix} \begin{matrix} 3 & -2 \\ 2 & -1 \\ 1 & 2 \end{matrix}$$

EXAMPLE 2 Use Diagonals

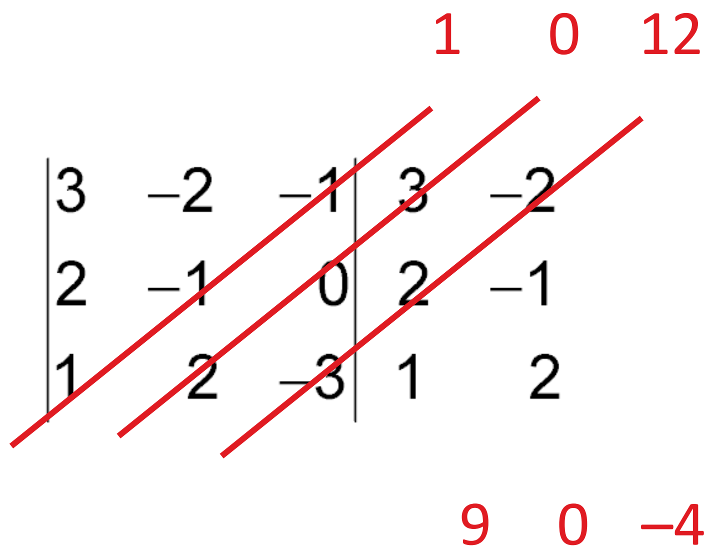
Step 2 Find the product of the elements of the diagonals.

$$
\left|\begin{array}{ccc} 3 & -2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{array}\right|\begin{array}{cc} 3 & -2 \\ 2 & -1 \\ 1 & 2 \end{array}
$$

$$9 \quad 0 \quad -4$$

EXAMPLE 2 Use Diagonals

Step 2 Find the product of the elements of the diagonals.



Step 3 Find the sum of each group.

$9 + 0 + (-4) = 5$ $\quad$ $1 + 0 + 12 = 13$

EXAMPLE 2 Use Diagonals

Step 4 Subtract the sum of the second group from the sum of the first group.

5 –13 = –8

Answer: The value of the determinant is –8.

Third – Order Determinant

Minors Rule

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - b_1 \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + c_1 \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$

EXAMPLE 2 **Use Minors**

$$\begin{vmatrix} 5 & 2 & 6 \\ 2 & -1 & 4 \\ -3 & 4 & 4 \end{vmatrix} = 5\begin{vmatrix} -1 & 4 \\ 4 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ -3 & 4 \end{vmatrix} + 6\begin{vmatrix} 2 & -1 \\ -3 & 4 \end{vmatrix}$$

$$= 5 \cdot (-4 - 16) - 2 \cdot (8 - -12) + 6 \cdot (8 - 3)$$

$$= -100 - 40 + 30$$

$$= -110$$

Key Concept — Identity Matrix for Multiplication

For Your FOLDABLE

Words The identity matrix for multiplication I is a square matrix with 1 for every element of the main diagonal, from upper left to lower right, and 0 in all other positions. For any square matrix A of the same dimension as I, $A \cdot I = I \cdot A = A$.

Symbols If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Key Concept

Inverse of a 2 × 2 Matrix

For Your FOLDABLE

The inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$.

Example 3 **Find the Inverse of a Matrix**

A. Find the inverse of the matrix, if it exists.

$$S = \begin{bmatrix} -1 & 0 \\ 8 & -2 \end{bmatrix}$$

Find the determinant.

$$\begin{bmatrix} -1 & 0 \\ 8 & -2 \end{bmatrix} = 2 - 0 = 2$$

Since the determinant is not equal to 0, S^{-1} exists.

Example 3 Find the Inverse of a Matrix

$$S^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Definition of inverse

$$= \frac{1}{-1(-2) - 8(0)}\begin{bmatrix} -2 & 0 \\ -8 & -1 \end{bmatrix}$$ $a = -1$, $b = 0$, $c = 8$, $d = -2$

$$= \frac{1}{2}\begin{bmatrix} -2 & 0 \\ -8 & -1 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & 0 \\ -4 & -\frac{1}{2} \end{bmatrix}$$ Simplify.

Answer: $\begin{bmatrix} -1 & 0 \\ -4 & -\frac{1}{2} \end{bmatrix}$

Example 3 Find the Inverse of a Matrix

B. Find the inverse of the matrix, if it exists.

$$T = \begin{bmatrix} -4 & 6 \\ -2 & 3 \end{bmatrix}$$

Find the value of the determinant.

$$\begin{bmatrix} -4 & 6 \\ -2 & 3 \end{bmatrix} = -12 + 12 = 0$$

Answer: Since the determinant equals 0, T^{-1} does not exist.

EXAMPLE 4

Solve a System of Equations

$x + y = 15$

$15x + 18y = 261$

The matrix equation is $\begin{bmatrix} 1 & 1 \\ 15 & 18 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 261 \end{bmatrix}$

coefficient matrix

variable matrix only the variables of a system

constant matrix only the constants of a system

EXAMPLE 4

Solve a System of Equations

STEP 1 Find the inverse of the coefficient matrix.

$$A^{-1} = \frac{1}{18-15}\begin{bmatrix} 18 & -1 \\ -15 & 1 \end{bmatrix} \text{ or } \frac{1}{3}\begin{bmatrix} 18 & -1 \\ -15 & 1 \end{bmatrix}$$

STEP 2 Multiply each side of the matrix equation by the inverse matrix.

$$\frac{1}{3}\begin{bmatrix} 18 & -1 \\ -15 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 1 \\ -15 & 18 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 18 & -1 \\ -15 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 261 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 36 \end{bmatrix}$$

EXAMPLE 4 Solve a System of Equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

The solution is (3, 12), where *x* represents the number of popcorn machines and *y* represents the number of water coolers.

Answer: The club rents 3 popcorn machines and 12 water coolers.

EXAMPLE 5

Two dump trucks have capacities of 12 and 18 tones. They make a total of 30 round trips to haul 474 tons of topsoil for a landscaping project. How many round trips does each truck make?

x = round trips made by 12 tons capacity truck

y = round trips made by 18 tons capacity truck

Step 1. Use the given information to set up a system of equations in two variables.

x + y = 30
12x + 18y = 474

Step 2. Write a matrix equation.

$$\begin{bmatrix} 1 & 1 \\ 12 & 18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 30 \\ 474 \end{bmatrix}$$

EXAMPLE 5

Step 3. Find inverse

$$\begin{bmatrix} 1 & 1 \\ 12 & 18 \end{bmatrix}^{-1} = \frac{1}{6} \cdot \begin{bmatrix} 18 & -1 \\ -12 & 1 \end{bmatrix}$$

Step 4. Multiply each of the matrix equation side by $\frac{1}{6} \cdot \begin{bmatrix} 18 & -1 \\ -12 & 1 \end{bmatrix}$

$$\frac{1}{6} \cdot \begin{bmatrix} 18 & -1 \\ -12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 12 & 18 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 18 & -1 \\ -12 & 1 \end{bmatrix} \cdot \begin{bmatrix} 30 \\ 474 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 66 \\ 114 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 19 \end{bmatrix}$$